System of linear equations

A **system of linear equations** (or a linear system) is a collection of one or more linear equations involving the same set of variables. Its general form is:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \multicolumn{9}{c}{\dotfill} \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array} \tag{1}$$

The matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called the **coefficient matrix**, and the matrix

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{12} & a_{22} & \cdots & a_{2n} & b_2 \\ \multicolumn{5}{c}{\dotfill} \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

which has one column more is called the **augmented matrix**.
A **solution** of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $(s_1, s_2, \ldots, s_n)$ are substituted for $(x_1, x_2, \ldots, x_n)$, respectively.
The solution set is the set of all possible solutions of a linear system.
Fact. A system of linear equations has either
i) exactly one solution (consistent) or
ii) infinitely many solutions (consistent) or
iii) no solution (inconsistent).
Elementary Row Operations:
1. (Replacement) Add one row to a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Each of those three operations has a restriction. Multiplying a row by 0 is not allowed because obviously that can change the solution set of the system. Similarly, adding a multiple of a row to itself is not allowed because adding $-1\times$row to itself gives the row of zeroes. Finally, swapping (i.e. interchange) a row with itself is disallowed to make some results in the next lectures easier to state and remember (and besides, self-swapping doesnt accomplish anything). The three elementary row operations are sometimes called the **Gaussian operations**.
Row equivalent matrices: Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.
Fact about Row Equivalence: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
Echelon form (or row echelon form):
1. All nonzero rows are above any rows of all zeros.
2. Each leading entry (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

Reduced echelon form: Add the following conditions to conditions 1, 2, and 3 above:
4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Theorem 1 *(Uniqueness of The Reduced Echelon Form) Each matrix is row-equivalent to one and only one reduced echelon matrix.*

Important Terms:
pivot position: a position of a leading entry in an echelon form of the matrix.
pivot: a nonzero number that either is used in a pivot position to create 0s or is changed into a leading 1, which in turn is used to create 0s.
pivot column: a column that contains a pivot position.
basic variable: any variable that corresponds to a pivot column in the augmented matrix of a system.
free variable: any nonbasic variable.

A system $\mathbf{Ax} = \mathbf{b}$ with m equations and n unknowns is defined by the $m \times n$ coefficient matrix $\mathbf{A}$ and the RHS vector $\mathbf{b}$.
The row reduced matrix rref($\mathbf{B}$) of the augmented matrix $\mathbf{B}$ determines the number of solutions of the system $\mathbf{Ax} = \mathbf{b}$. There are three possibilities:
Consistent. Exactly one solution. There is a leading 1 in each row but none in the last column of $\mathbf{B}$.
Inconsistent. No solutions. There is a leading 1 in the last column of $\mathbf{B}$.
Infinitely many solutions. There are rows of $\mathbf{B}$ without leading 1.
If $m < n$ (less equations then unknowns), then there are either zero or infinitely many solutions.
The rank$\mathbf{A}$ of a matrix $\mathbf{A}$ is the number of leading ones in rref($\mathbf{A}$).
Leading variables. The variables corresponding to the columns with leading ones in the reduced row echelon form of an augmented matrix are called leading variables. The other variables are called non-leading variables.
Three equivalent ways of viewing a linear system:
1. as a system of linear equations;
2. as a vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$ or
3. as a matrix equation $\mathbf{Ax} = \mathbf{b}$.